Large N Matrix Models
and
q-Deformed Quantum Field Theories

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Abstract

Recently it was shown that an asymptotic behaviour of $SU(N)$ gauge theory for large N is described by q-deformed quantum field. The master fields for large N theories satisfy to standard equations of relativistic field theory but fields satisfy q-deformed commutation relations with $q = 0$. These commutation relations are realized in the Boltzmannian Fock space. The master field for gauge theory does not take values in a finite-dimensional Lie algebra however there is a non-Abelian gauge symmetry. The gauge master field for a subclass of planar diagrams, so called half-planar diagrams, is also considered. A recursive set of master fields summing up a matreoshka of 2-particles reducible planar diagrams is briefly described.

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1   Introduction

With great pleasure I dedicate my paper to Jurek Lukierski. He made a great contribution in quantum field theory. I hope that this paper will be interesting for him since his last interests are related with q-deformed algebras [1, 2].

In the last years there were considerations of models of quantum field theory based on quantum groups and q-deformed commutation relations [1]-[20]. The main problem with these approaches was how to find a physical justification for q-deformed quantum field theory. Recently, it was shown that there is a remarkable physical justification for q-deformed quantum field theory at least for $q = 0$. In [21] it was shown that such theory describes an asymptotic behaviour of $SU(N)$ gauge theory for $N \to \infty$.

The large N limit in QCD where N is the number of colours enables us to understand qualitatively certain striking phenomenological features of strong interactions [22]-[25]. To perform an analytical investigation one needs to compute the sum of all planar diagrams. Summation of planar diagrams has been performed only in low dimensional space-time [26, 27].

It was suggested [24] that there exists a master field which dominates the large N limit. There was an old problem in quantum field theory how to construct the master field for the large N limit in QCD. This problem has been discussed in many works, see for example [29]-[35]. More recently the problem has been reconsidered [36]-[46] by using methods of non-commutative (quantum) probability theory [47, 48, 41]. Gopakumar and Gross [36] and Douglas [37] have described the master field using a knowledge of all correlation functions of a model.

Finally the problem of construction of the master field has been solved in [21]. It was shown that the master field satisfies to standard equations of relativistic field theory but it is quantized according to q-deformed ($q = 0$) relations

$$a(k)a^*(k') = \delta(k - k'),$$

where $a(k)$ and $a^*(k)$ are annihilation and creation operators. These operators have a realization in the free (Boltzmannian) Fock space. Therefore, to study the large N limit of QCD it seems reasonable to develop methods of treatment of theory in the Boltzmann space. Let us note that the fact that the master field satisfies the same equations as usual relativistic fields makes actual a development of non-perturbative methods of investigation of these equations.

Quantum field theory in Boltzmannian Fock space has been considered in [43]-[46]. Some special form of this theory realizes the master field for a subset of planar diagrams, for the so called half-planar (HP) diagrams and gives an analytical summations of HP diagrams [43]-[46]. In this paper a realization for the master field for HP diagrams of gauge theories will be given. Then some recursive set of master fields summing up a more rich subclass of planar diagrams will be also sketched. This subclass of diagrams contains a matreoshka of 2-particles reducible planar diagrams.

In Section 2 consider the master field for matrix models including the master field for gauge field will be considered. In Section 3 Boltzmann quantum field theory and the HP master field for scalar and gauge theories will be considered.

2   Master Field for Planar Diagrams

2.1 Zero-dimensional Master Field

The master field Φ for the Gaussian matrix model in zero dimensional space-time is defined by the relation

$$\lim_{N\to\infty} \frac{1}{Z_N} \int \frac{1}{N^{1+k/2}} \operatorname{tr} M^k e^{-S(M)} dM = <0|\Phi^k|0> \tag{2.1}$$

for $k = 1, 2...$, where the action $S(M) = \frac{1}{2} \operatorname{tr} M^2$ and M is an Hermitian $N \times N$ matrix.

The operator

$$\Phi = a + a^*$$

acts in the free or Boltzmannian Fock space over C with vacuum vector $|0>$

$$a|0> = 0 \tag{2.2}$$

and creation and annihilation operators satisfying the relation

$$aa^* = 1 \tag{2.3}$$

Let us recall that the free (or Boltzmannian) Fock space $F(H)$ over the Hilbert space H is the tensor algebra over H

$$\mathcal{T}(H) = \oplus_{n=0}^{\infty} H^{\otimes n}.$$

Creation and annihilation operators are defined as

$$a^*(f) f_1 \otimes ... \otimes f_n = f \otimes f_1 \otimes ... \otimes f_n$$

$$a(f) f_1 \otimes ... \otimes f_n = <f, f_1> \otimes f_2 \otimes ... \otimes f_n$$

where $<f, g>$ is the inner product in H. One has

$$a(f) a^*(g) = <f, g>.$$

Here we consider the simplest case $H = C$.

The relation (2.1) has been obtained in physical [29] and mathematical [47] works. It can be interpreted as a central limit theorem in non-commutative (quantum) probability theory, for a review see [41]. The basic notion of non-commutative probability theory is an algebraic probability space, i.e a pair (A, h) where A is an algebra and h is a positive linear functional on A. An example of the algebraic probability space is given by the algebra of random matrix with (2.1) being non-commutative central limit theorem. As another example one can consider quantum groups. Theory of quantum groups have received in the last years a lot of attention [49]-[52]. In this case A is the Hopf algebra of functions on the quantum group and h is the quantum Haar measure.

The relations of theory of the master field in the Boltzmannian Fock space with quantum groups was discussed in [42]. There we defined the master field algebra and showed that this algebra is isomorphic to the algebra of functions on the quantum semigroup $SU_q(2)$ for $q = 0$. In fact the master field algebra coincides with the algebra of the so called central elements of the quantum group Hopf algebra. Let us repeat the main steps of these observations. In the Boltzmannian Fock space F(C) we have

$$a^*a = 1 - |0><0|. \tag{2.4}$$

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Let us define an operator
$$F = e^{i\phi}|0 >< 0|, \tag{2.5}$$

where ϕ is an arbitrary real number. Then from (2.2), (2.4), (2.3) one has the following relations:
$$aF = 0, \qquad aF^* = 0, \qquad FF^* = F^*F,$$
$$aa^* = 1, \quad a^*a + FF^* = 1. \tag{2.6}$$

We call the algebra (2.6) the master field algebra. ¿From equations (2.6) we get

$$(FF^*)^2 = FF^*$$

and the operator FF^* is an orthogonal projector.

Now let us recall the definition of algebra of functions $A_q = Fun(SU_q(2))$ on the quantum group $SU_q(2)$. The algebra A_q is the Hopf algebra with generators a, a^*, c, c^* satisfying the relations

$$ac^* = qc^*a, \qquad ac = qca, \qquad cc^* = c^*c,$$
$$a^*a + cc^* = 1, \quad aa^* + q^2cc^* = 1 \tag{2.7}$$

where $0 < |q| < 1$. Taking $q = 0$ in (2.7) one gets the relations (2.6) if $F = c$. Therefore the master field algebra (2.6) is isomorphic to the algebra A_0 of functions on the quantum (semi) group $SU_q(2)$ for $q = 0$.

A_q is a Hopf algebra with the standard coproduct,

$$\Delta : A_q \to A_q \otimes A_q$$

$$\Delta(g_j^i) = \sum_{k=0,1} g_k^i \otimes g_j^k.$$

The Boltzman field Φ is a central element of the Hopf algebra A_q.

The bosonization of the quantum group $SU_q(2)$ [20] gives bosonization for the master field. If b and b^* are the standard creation and annihilation operators in the Bosonic Fock space,
$$[b, b^*] = 1, \quad b|0 >= 0,$$

then
$$a = \sqrt{\frac{1 - q^{2(N+1)}}{N + 1}}b, \quad c = e^{i\phi}q^N \tag{2.8}$$

satisfies the relations (2.7). Here $N = b^*b, \phi$ is a real number. If $q \to 0$ one gets from (2.8)
$$a = \frac{1}{\sqrt{N + 1}}b, \quad c = e^{i\phi}|0 >< 0|. \tag{2.9}$$

Therefore the master field takes the form

$$\Phi = b^* \frac{1}{\sqrt{N + 1}} + \frac{1}{\sqrt{N + 1}}b. \tag{2.10}$$

The operator N can be also written in terms of creation and annihilation operators a^+, a, $N = \sum_{k=1}^{\infty}(a^+)^k(a)^k$.

2.2 Master Field as a Classical Matrix

Let us consider $U(N)$ invariant correlation functions for a model of selfinteracting Hermitian scalar matrix field $M(x) = (M_{ij}(x))$, $i, j = 1, ..., N$ in the D-dimensional Euclidean space-time

$$< \text{tr}\,(\,M(x_1)...M(x_k))\,> = \frac{1}{Z_N} \int \text{tr}\,(\,M(x_1)...M(x_k))e^{-S(M)}dM \qquad (2.11)$$

where $S(M)$ is the action

$$S(M) = \int d^D x [\frac{1}{2} \text{tr}\,(\,\partial M)^2 + \sum \frac{c_i}{N^{-1+i/2}} \text{tr}\, M^{\,i}]$$

and M is an Hermitian $N \times N$ matrix,

$$Z_N = \int e^{-S(M)}dM \qquad (2.12)$$

Witten suggested [24] that there exists a master field which dominates in the large N limit of invariant correlation functions of a matrix field, i.e.

$$\lim_{N\to\infty} \frac{1}{Z_N} \int \frac{1}{N^{1+k/2}} \text{tr}\,(\,M(x_1)...M(x_k))e^{-S(M)}dM \,=\, \text{tr}\,(\,\mathcal{M}(x_1)...\mathcal{M}(x_k)), \qquad (2.13)$$

where $\mathcal{M}$ is some $\infty \times \infty$ matrix. Since $\infty \times \infty$ matrix can be considered as an operator acting in an infinite dimension space one can interpret the RHS of (2.13) as an expectation value of the product of some operators $\Phi(x_i)$

$$\text{tr}\,(\,\mathcal{M}(x_1)...\mathcal{M}(x_k)) = <\Phi(x_n)...\Phi(x_1)> \qquad (2.14)$$

This interpretation gives an alternative definition of the master field $\Phi(x)$ as a scalar operator which realizes the following relation

$$\lim_{N\to\infty} \frac{1}{N^{1+k/2}} < \text{tr}\, M\,(x_1)...M(x_k) > = <\Phi(x_n)...\Phi(x_1)> \qquad (2.15)$$

where $< . >$ means some expectation value. Therefore the problem is in constructing of a scalar field Φ acting in some space so that the expectation value of this field reproduces the large N asymptotic of $U(N)$ invariant correlation functions of given matrix field.

2.3 Free Master Field

To construct master field for the free matrix field let us calculate the expectation value for free matrix field in the Euclidean space-time

$$\frac{1}{N^3} < \text{tr}\,(\,M(x_1)M(x_2)M(x_3)M(x_4))\,>^{(0)} = \qquad (2.16)$$

$$\frac{1}{Z_N} \int \frac{1}{N^3} \text{tr}\,(\,M(x_1)M(x_2)M(x_3)M(x_4))e^{-S_0(M)}dM$$

where the action

$$S_0(M) = \int d^D x [\frac{1}{2} \text{tr}\,(\,\partial M)^2].$$

We have

$$\frac{1}{N^3} < \text{tr} \, (\, M(x_1)M(x_2)M(x_3)M(x_4)) \, >^{(0)} = D(x_1 - x_2)D(x_3 - x_4) + \tag{2.17}$$

$$D(x_1 - x_4)D(x_2 - x_3) + \frac{1}{N}D(x_1 - x_3)D(x_2 - x_4),$$

here we use

$$< M_{ij}(x)M_{j'i'}(y) >^{(0)} = \delta_{ii'}\delta_{jj'}D(x - y), \tag{2.18}$$

D is an Euclidean propagator,

$$D(x - y) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik(x-y)}}{k^2 + m^2} \tag{2.19}$$

Let

$$\phi(x) = \phi^+(x) + \phi^-(x) \tag{2.20}$$

be the Bolzmann field with creation and annihilation operators satisfying the relations

$$\phi^-(x)\phi^+(y) = D(x - y), \tag{2.21}$$

It is easy to check that

$$< 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0 > = D(x_1 - x_2)D(x_3 - x_4) + D(x_1 - x_4)D(x_2 - x_3) \tag{2.22}$$

where $|0>$ is a vacuum $\phi^-(x)|0> = 0 = < 0|\phi^+(x)$. The similar relation is true for an arbitrary n-point correlation function. This consideration proves that the Euclidean Boltzmann field is a master field for the Euclidean free matrix model.

Moreover, if we assume the relations

$$\phi_q^-(x)\phi_q^+(y) + q\phi_q^+(y)\phi_q^-(x) = D(x - y), \tag{2.23}$$

then we get

$$< 0|\phi_q(x_1)\phi_q(x_2)\phi_q(x_3)\phi_q(x_4)|0 > = \tag{2.24}$$

$$D(x_1 - x_2)D(x_3 - x_4) + D(x_1 - x_4)D(x_2 - x_3) + qD(x_1 - x_3)D(x_2 - x_4),$$

i.e. relation (2.24) reproduces (2.17) if we identify

$$\frac{1}{N} = q. \tag{2.25}$$

We can also consider the Minkowski space time. To avoid misunderstanding we use a notation $M^{(in)}$ for the free Minkowski matrix field. One has

$$< 0|M_{ij}^{(in)}(x)M_{pq}^{(in)}(y)|0 > = \delta_{iq}\delta_{jp}D^-(x - y) \tag{2.26}$$

where

$$D^-(x) = \frac{1}{(2\pi)^3} \int e^{ikx}\theta(-k^0)\delta(k^2 - m^2)dk.$$

Let consider the free scalar Boltzmannian field $\phi^{(in)}(x)$ given by

$$\phi^{(in)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{\sqrt{2\omega(k)}}(a^*(k)e^{ikx} + a(k)e^{-ikx}), \tag{2.27}$$

where $\omega(k) = \sqrt{k^2 + m^2}$. It satisfies to the Klein-Gordon equation

$$(\Box + m^2)\phi^{(in)}(x) = 0$$

and it is an operator in the Boltzmannian Fock space with relations

$$a(k)a^*(k') = \delta^{(3)}(k - k') \tag{2.28}$$

and vacuum $|\Omega_0)$, $a(k)|\Omega_0) = 0$. A systematical consideration of the Wightman formalism for Boltzmannian fields is presented in [44].

$a(k)$ and $a^*(k)$ act in the Boltzmannian Fock space $\Gamma(H)$ over $H = L^2(R^3)$ and one uses notations such as $a(f) = \int a(k)f(k)dk$. An n-particle state is created from the vacuum $|\Omega_0) = 1$ by the usual formula

$$|k_1, ..., k_n) = a^*(k_1)...a^*(k_n)|\Omega_0)$$

but it is not symmetric under permutation of k_i.

The the following basic relation takes place

$$\lim_{N \to \infty} \frac{1}{N^{1+\frac{k}{2}}} < 0| \, \mathrm{tr} \, (\, (M^{(in)}(y_1))^{p_1}...(M^{(in)}(y_r))^{p_r})|0 > \tag{2.29}$$

$$= (\Omega_0|(\phi^{(in)}(y_1))^{p_1}...(\phi^{(in)}(x_r))^{p_r}|\Omega_0)$$

where $k = p_1 + ... + p_r$. To prove (2.29) one uses the Wick theorem for the Wightman functions and 't Hooft's graphs with double lines. According to the Wick theorem we represent the vacuum expectation value in the L.H.S. of (2.29) as a sum of 't Hooft's graphs with the propagators (2.26). Then in the limit $N \to \infty$ only non-crossing (rainbow) graphs are nonvanished. We get the same expression if we compute the R.H.S. of (2.29) by using the relations (2.28), i.e. by using the Boltzmannian Wick theorem.

2.4   Master Field for Interacting Matrix Scalar Field

To construct the master field for interacting quantum field theory [21] we have to work in Minkowski space-time and use the Yang-Feldman formalism [53]-[56]. Let us consider a model of an Hermitian scalar matrix field $M(x) = (M_{ij}(x))$, $i, j = 1, ..., N$ in the 4-dimensional Minkowski space-time with the field equations

$$(\Box + m^2)M(x) = J(x) \tag{2.30}$$

We take the current $J(x)$ equal to

$$J(x) = -\frac{g}{N}M^3(x) \tag{2.31}$$

where g is the coupling constant but one can take a more general polynomial over $M(x)$. One integrates eq (2.30) to get the Yang-Feldman equation [53, 54]

$$M(x) = M^{(in)}(x) + \int D^{ret}(x - y)J(y)dy \tag{2.32}$$

where $D^{ret}(x)$ is the retarded Green function for the Klein-Gordon equation,

$$D^{ret}(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{m^2 - k^2 - i\epsilon k^0} dk$$

and $M^{(in)}(x)$ is a free Bose field. The $U(N)$-invariant Wightman functions are defined as

$$W(x_1, ..., x_k) = \frac{1}{N^{1+\frac{k}{2}}} < 0| \operatorname{tr} (M(x_1)...M(x_k))|0 > \qquad (2.33)$$

where $|0>$ is the Fock vacuum for the free field $M^{(in)}(x)$.

We will show that the limit of functions (2.33) when $N \to \infty$ can be expressed in terms of a quantum field $\phi(x)$ (the master field) which is a solution of the equation

$$\phi(x) = \phi^{(in)}(x) + \int D^{ret}(x - y)j(y)dy \qquad (2.34)$$

where

$$j(x) = -g\phi^3(x) \qquad (2.35)$$

The master field $\phi(x)$ does not have matrix indexes.

The following theorem is true.

Theorem 1. *At every order of perturbation theory in the coupling constant one has the following relation*

$$\lim_{N \to \infty} \frac{1}{N^{1+\frac{k}{2}}} < 0| \operatorname{tr} (M(x_1)...M(x_k))|0 >= (\Omega_0|\phi(x_1)...\phi(x_k)|\Omega_0) \qquad (2.36)$$

where the field $M(x)$ is defined by (2.32) and $\phi(x)$ is defined by (2.34).

The proof of the theorem see in [21].

2.5   Gauge field

In this section we construct the master field for gauge field theory. Let us consider the Lagrangian

$$L = \operatorname{tr} \{ -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + \bar{c}\partial_\mu \nabla_\mu c\} \qquad (2.37)$$

where A_μ is the gauge field for the $SU(N)$ group, c and $\bar{c}$ are the Faddeev-Popov ghost fields and α is a gauge fixing parameter. The fields A_μ, c and $\bar{c}$ take values in the adjoint representation. Here

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{g}{N^{\frac{1}{2}}}[A_\mu, A_\nu], \quad \nabla_\mu c = \partial_\mu c + \frac{g}{N^{\frac{1}{2}}}[A_\mu, c],$$

g is the coupling constant. Equations of motion have the form

$$\nabla_\mu F_{\mu\nu} + \frac{1}{\alpha}\partial_\nu\partial_\mu A_\mu + \frac{g}{N^{\frac{1}{2}}}\partial_\nu\bar{c}c + \frac{g}{N^{\frac{1}{2}}}c\partial_\nu\bar{c} = 0, \qquad (2.38)$$

$$\partial_\mu(\nabla_\mu c) = 0, \quad \nabla_\mu(\partial_\mu\bar{c}) = 0$$

One writes these equations in the form

$$\Box A_\nu - (1 - \frac{1}{\alpha})\partial_\nu\partial_\mu A_\mu = J_\nu, \qquad (2.39)$$

$$\Box c = J, \quad \Box \bar{c} = \bar{J},$$

where

$$J_\nu = -\frac{g}{N^{\frac{1}{2}}}\partial_\mu[A_\mu, A_\nu] - \frac{g}{N^{\frac{1}{2}}}[A_\mu, F_{\mu\nu}] - \frac{g}{N^{\frac{1}{2}}}\partial_\nu\bar{c}c - \frac{g}{N^{\frac{1}{2}}}c\partial_\nu\bar{c},$$

$$J = -\frac{g}{N^{\frac{1}{2}}}\partial_\mu[A_\mu, c], \quad \bar{J} = -\frac{g}{N^{\frac{1}{2}}}[A_\mu, \partial_\mu\bar{c}]$$

¿From (2.39) one gets the Yang-Feldman equations

$$A_\mu(x) = A_\mu^{(in)}(x) + \int D_{\mu\nu}^{ret}(x-y)J_\nu(y)dy, \tag{2.40}$$

$$c(x) = c^{(in)}(x) + \int D^{ret}(x-y)J(y)dy, \quad \bar{c}(x) = \bar{c}^{(in)}(x) + \int D^{ret}(x-y)\bar{J}(y)dy,$$

where

$$D_{\mu\nu}^{ret}(x) = (g_{\mu\nu} - (1-\alpha)\frac{\partial_\mu\partial_\nu}{\Box})D^{ret}(x),$$

and $g_{\mu\nu}$ is the Minkowski metric. Free in-fields satisfy

$$(\Box g_{\mu\nu} - (1 - \frac{1}{\alpha})\partial_\mu\partial_\nu)A_\nu^{(in)}(x) = 0,$$

$$\Box c^{(in)}(x) = 0, \quad \Box\bar{c}^{(in)}(x) = 0.$$

and they are quantized in the Fock space with vacuum $|0>$. The vector field $A_\mu^{(in)}$ is a Bose field and the ghost fields $c^{(in)}, \bar{c}^{(in)}$ are Fermi fields. Actually one assumes a gauge $\alpha = 1$. In a different gauge one has to introduce additional ghost fields. We introduce the notation $\psi_i = (A_\mu, c, \bar{c})$ for the multiplet of gauge and ghost fields. The $U(N)$-invariant Wightman functions are defined as

$$W(x_1, ..., x_k) = \frac{1}{N^{1+\frac{k}{2}}} < 0|tr(\psi_{i_1}(x_1)...\psi_{i_k}(x_k))|0 > . \tag{2.41}$$

We will show that the limit of functions (2.41) when $N \to \infty$ can be expressed in terms of the master fields. The master field for the gauge field $A_\mu(x)$ we denote $B_\mu(x)$ and the master fields for the ghost fields $c(x), \bar{c}(x)$ will be denoted $\eta(x), \bar{\eta}(x)$. The master fields satisfy to equations

$$D_\mu\mathcal{F}_{\mu\nu} + \frac{1}{\alpha}\partial_\nu\partial_\mu B_\mu + g\partial_\nu\bar{\eta}\eta + g\eta\partial_\nu\bar{\eta} = 0,$$

$$\partial_\mu(D_\mu\eta) = 0, \quad D_\mu(\partial_\mu\bar{\eta}) = 0 \tag{2.42}$$

where

$$\mathcal{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + g[B_\mu, B_\nu], \quad D_\mu\eta = \partial_\mu\eta + g[B_\mu, \eta]. \tag{2.43}$$

These equations have the form of the Yang-Mills equations (2.39) however the master fields B_μ, η, $\bar{\eta}$ do not have matrix indexes and they do not take values in a finite dimensional Lie algebra. The gauge group for the field B_μ is an infinite dimensional group of unitary operators in the Boltzmannian Fock space. Equations (2.42) in terms of currents read

$$\Box B_\nu - (1 - \frac{1}{\alpha})\partial_\nu\partial_\mu A_\mu = j_\nu, \tag{2.44}$$

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$$\Box \eta = j, \quad \Box \bar{\eta} = \bar{j},$$

where

$$j_\nu = -g\partial_\mu[B_\mu, B_\nu] - g[B_\mu, \mathcal{F}_{\mu\nu}] - g\partial_\nu\bar{\eta}\eta - g\eta\partial_\nu\bar{\eta},$$

$$j = -g\partial_\mu[B_\mu, \eta], \quad \bar{j} = -g[B_\mu, \partial_\mu\bar{\eta}].$$

We define the master fields by using the Yang-Feldman equations

$$B_\mu(x) = B_\mu^{(in)}(x) + \int D_{\mu\nu}^{ret}(x - y)j_\nu(y)dy, \tag{2.45}$$

$$\eta(x) = \eta^{(in)}(x) + \int D^{ret}(x - y)j(y)dy, \quad \bar{\eta}(x) = \bar{\eta}^{(in)}(x) + \int D^{ret}(x - y)\bar{j}(y)dy,$$

The in-master fields are quantized in the Boltzmannian Fock space. For the master gauge field we have

$$B_\mu^{(in)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2|k|}} \sum_{\lambda=1}^{4} \epsilon_\mu^{(\lambda)}(k)[a^{(\lambda)*}(k)e^{ikx} + a^{(\lambda)}(k)e^{-ikx}), \tag{2.46}$$

where $\epsilon_\mu^{(\lambda)}(k)$ are polarization vectors and annihilation and creation operators satisfy

$$a^{(\lambda)}(k)a^{(\lambda')*}(k') = g^{\lambda\lambda'}\delta^{(3)}(k - k'), \tag{2.47}$$

The expression (2.46) for the field $B_\mu(x)$ looks like an expression for the photon field. However because of relations (2.47) the commutator $[B_\mu(x), B_\nu(x)]$ does not vanish and it permits us to develop a gauge theory for the field $B_\mu(x)$ with a non-Abelian gauge symmetry.

We quantize the master ghost fields in the Boltzmannian Fock space with indefinite metric

$$\eta^{(in)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2|k|}} (\gamma^*(k)e^{ikx} + \beta(k)e^{-ikx}), \tag{2.48}$$

$$\bar{\eta}^{(in)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2|k|}} (\beta^*(k)e^{ikx} + \gamma(k)e^{-ikx}),$$

where creation and annihilation operators satisfy

$$\gamma(k)\gamma^*(k') = \delta^{(3)}(k - k'),$$

$$\beta(k)\beta^*(k') = -\delta^{(3)}(k - k'). \tag{2.49}$$

We also assume that the product of any annihilation operator with a creation operator of a different type always is equal to zero, i.e.

$$\gamma(k)\beta^*(k') = \beta(k)\gamma^*(k') = a^{(\lambda)}(k)\gamma^*(k) = 0,$$

$$a^{(\lambda)}(k)\beta^*(k') = \gamma(k)a^{(\lambda)*}(k') = \beta(k)a^{(\lambda)*}(k') = 0. \tag{2.50}$$

The Boltzmannian Fock vacuum satisfies

$$\gamma(k)|\Omega_0) = \beta(k)|\Omega_0) = a^{(\lambda)}(k)|\Omega_0) = 0. \tag{2.51}$$

Let us denote $\chi_i = (B_\mu, \eta, \bar{\eta})$ the multiplet of the master fields. The following theorem is true.

Theorem 2. *At every order of perturbation theory in the coupling constant one has the following relation*

$$\lim_{N \to \infty} \frac{1}{N^{1+\frac{k}{2}}} < 0| \operatorname{tr} (\psi_{i_1}(x_1)...\psi_{i_k}(x_k))|0 >= (\Omega_0|\chi_{i_1}(x_1)...\chi_{i_k}(x_k)|\Omega_0) \qquad (2.52)$$

where the fields $A_\mu(x), c(x)$ and $\bar{c}(x)$ are defined by (2.40) and $B_\mu(x), \eta(x)$ and $\bar{\eta}(x)$ are defined by (2.45).

The proof of Theorem 2 is analogous to the proof of Theorem 1. We get relations (2.49) for master fields by taking into account the wrong statistics of the ghost fields.

3 Master Field for HP Diagrams

3.1 Half-Planar Approximation for the One Matrix Model

A free n-point Green's function is defined as the vacuum expectation of n-th power of master field

$$G_n^{(0)} = \langle 0|\phi^n|0\rangle. \qquad (3.1)$$

As it is well-known, the Green's function (3.1) is given by a n-th moment of Wigner's distribution [26, 47]

$$G_{2n}^{(0)} = \int_{-2}^{2} \frac{d\lambda}{2\pi} \lambda^{2n} \sqrt{4 - \lambda^2} = \frac{(2n)!}{n!(n+1)!}.$$

This representation can be also obtained as a solution of the Schwinger-Dyson equations

$$G_{2n}^{(0)} = \sum_{m=1}^{n} G_{2m-2}^{(0)} G_{2n-2m}^{(0)}.$$

Interacting Boltzmann correlation functions are defined by the formula [43]

$$G_n = \langle 0|\phi^n(1 + S_{int}(\phi))^{-1}|0\rangle. \qquad (3.2)$$

In contrast to the ordinary quantum field theory where one deals with the exponential function of an interaction, here we deal with the rational function of an interaction. In [45] it was shown that under natural assumptions the form (3.2) is unique one which admits Schwinger-Dyson-like equations.

For the case of quartic interaction $S_{int} = g\phi^4$ the Boltzmannian Schwinger-Dyson equations have the form

$$G_n = \sum_{l=1}^{k-1} G_{k-l-1}^{(0)} G_{l+n-k-1} + \sum_{l=k+1}^{n} G_{l-k-1}^{(0)} G_{n+k-l-1} \qquad (3.3)$$

$$-g[G_{n-k}G_{k+2} + G_{n-k+1}G_{k+1} + G_{n-k+2}G_k + G_{n-k+3}G_{+k-1}].$$

For 2- and 4-point correlation functions we have

$$G_2 = 1 - gG_2G_2 - gG_4, \quad G_4 = 2G_2 - 2gG_2G_4. \qquad (3.4)$$

3.2 Boltzmann Correlation Functions for D-Dimensional Space-Time

Here we present the Schwinger-Dyson equations for Boltzmann correlation functions in D-dimensional Euclidean space. To avoid problems with tadpoles let us following [46] consider the two-field formulation. We adopt the following notations. Let $\psi(x) = \psi^+(x) + \psi^-(x)$, $\phi(x) = \phi^+(x) + \phi^-(x)$ be the Bolzmann fields with creation and annihilation operators satisfying the relations

$$\psi^-(x)\psi^+(y) = \phi^-(x)\phi^+(y) = D(x,y),$$

$$\psi^-(x)\phi^+(y) = \phi^-(x)\psi^+(y) = 0,$$

where $D(x,y)$ is D-dimensional Euclidean propagator. The n-point Green's function is defined by

$$F_n(x_1,...,x_n) = \langle 0|\psi(x_1)\phi(x_2)...\phi(x_{n-1})\psi(x_n)(1 + \int d^D x g\psi : \phi\phi : \psi))^{-1}|0\rangle. \tag{3.5}$$

We define an one-particle irreducible (1PI) 4-point function $\Gamma_4(x,y,z,t)$ as

$$\Gamma_4(x,y,z,t) = \int dx'dy'dz'dt' F_2^{-1}(x,x')D^{-1}(y,y')D^{-1}(z,z') \times \tag{3.6}$$

$$F_2^{-1}(t,t')\mathcal{F}_4(x',y',z',t'),$$

where $\mathcal{F}_4$ is a connected part of F_4

$$F_4(x,y,z,t) = \mathcal{F}_4(x,y,z,t) + F_2(x,t)D(y,z).$$

Note that in the contrast to the usual case in the RHS of (3.6) we multiply $\mathcal{F}_4$ only on two full 2-point Green functions while in the usual case to get an 1PI Green function one multiplies an n-point Green function on n full 2-point functions.

Let us write down the Schwinger-Dyson equations for the two- and four-point correlation functions. We have

$$\Gamma_4(p,k,r) = -g - g \int dk' F_2(p+k-k')D(k')\Gamma_4(p+k-k',k',r) \tag{3.7}$$

$$\Sigma_2(p) = g \int dkdq F_2(k)D(q)D(p-k-q)\Gamma_4(p,k,q). \tag{3.8}$$

where

$$F_2 = \frac{1}{p^2 + m^2 + \Sigma_2}$$

Equation (3.7) is the Bethe-Salpeter-like equation with the kernel which contains an unknown function F_2. Equation (3.8) is similar to the usual relation between the self-energy function Σ_2 and the 4-point vertex function for φ^4 field theory, meanwhile equation (3.7) is specific for the Boltzmann field theory. A special approximation reduces this system of integral equations to a linear integral equation which was considered [58] in the rainbow approximation in the usual field theory.

3.3   A Matreoshka of 2-particles Reducible Diagrams

In [43] has been shown that equations sum up HP diagrams of planar theory. Let us remind the definition of HP diagrams. Sometimes they are called the rainbow diagrams. The free rainbow diagrams are dual to tree diagrams and they have been summed up in the zero dimensions [57]. The half-planar diagrams for $< \mathrm{tr}\,(\,M(x_n),...M(x_n)) >$ are defined as a part of planar non-vacuum diagrams which are topologicaly equivalent to the graphs with all vertexes lying on some plane line in the left of generalized vertex represented $\mathrm{tr}\,(\,M(x_n),...M(x_n))$ and all propagators lying in the half plane.

We can use F_2 and F_4 to construct correlations functions which correspond the sum of more complicate diagrams. Let us consider the following correlations functions

$$F_n^{(1)}(x_1, ..., x_n) = \langle 0|\psi^{(1)}(x_1)\phi(x_2)...\phi(x_{n-1})\psi^{(1)}(x_n)$$ (3.9)

$$(1 + \int \prod_{i=1}^{i=4} d^D x_i \Gamma_4(x_2, x_3, x_4, x_1)\psi^1(x_1) : \phi(x_2)\phi(x_3) : \psi^1(x_4)))^{-1}|0\rangle,$$

here

$$\psi^{(1)-}(x)\psi^{(1)+}(y) = F_2(x, y)$$ (3.10)

The Schwinger-Dyson equations for the 2- and 4-point correlation functions $F_2^{(1)}$ and $F_4^{(1)}$ satisfy to equations similar to equations (3.7) and (3.8). The obtained $F_2^{(1)}$ and $F_4^{(1)}$ may be used to define the next approximation to planar diagrams. One can see that such procedure sums up special type of 2-particles reducible diagrams. These diagrams are specified by the property that they contain two lines so that after removing these lines from the given diagrams one reminds with two disconnected parts and each of these disconnected part is itself a connected 2-particle reducible diagram. It is natural to call this set as an matreoshka of 2-particle reducible diagrams.

3.4   Master Field for HP Gauge Theory

In the case of gauge theory the set of the HP master field is given by the field satisfying the following relations

$$A_\mu^-(x)A_\mu^+(y) = \frac{1}{(2\pi)^D}\int d^D k (g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2})\frac{1}{k^2}\exp ik(x-y),$$ (3.11)

$$\bar{c}^-(x)c^+(y) = \frac{1}{(2\pi)^D}\int d^D k \frac{1}{k^2}\exp ik(x-y),$$ (3.12)

$$c^-(x)\bar{c}^+(y) = -\frac{1}{(2\pi)^D}\int d^D k \frac{1}{k^2}\exp ik(x-y),$$ (3.13)

$$A_\mu^-(x)c^+(y) = A_\mu^-(x)\bar{c}^+(y) = 0$$ (3.14)

A unrenormalized interacting Lagrangian which assumed to enter in the correlation functions as is (3.5) (a generalization to two-fields formalism is evident) has the form

$$L_{unr} = \frac{g}{4}\{\ \partial_\nu A_\mu[A_\mu, A_\nu] + A_\nu\partial_\nu A_\mu A_\mu - A_\mu\partial_\nu A_\mu A_\nu + [A_\nu, A_\mu]\partial_\nu A_\mu\}+$$

$$4g^2\{2A_\nu A_\mu A_\nu A_\mu - A_\nu A_\mu A_\mu A_\nu - A_\mu A_\mu A_\nu A_\nu\}$$

$$-g\{\partial_\mu\bar{c}A_\mu c + c\partial_\mu\bar{c}A_\mu + A_\mu c\partial_\mu\bar{c} - \partial_\mu\bar{c}cA_\mu - A_\mu\partial_\mu\bar{c}c - cA_\mu\partial_\mu\bar{c}\} \tag{3.15}$$

Divergences in one-loop correlation functions may be removed by the following renormalizations

$$L_r = \frac{g}{4}Z_1^{HP}(Z_2^{HP})^{-1}\{\partial_\nu A_\mu[A_\mu, A_\nu] + +A_\nu\partial_\nu A_\mu A_\mu - A_\mu\partial_\nu A_\mu A_\nu + [A_\nu, A_{mu}]\partial_\nu A_\mu\} +$$

$$+\frac{g^2}{2}Z_4^{HP}(Z_2^{HP})^{-1}\{2A_\nu A_{mu}A_\nu A_{mu} - A_\nu A_{mu}A_\mu A_\nu - A_\mu A_\mu A_\nu A_\nu\} -$$

$$\frac{g}{8}\tilde{Z}_1^{HP}(\tilde{Z}_2^{HP})^{-1}\partial_\mu\bar{c}A_\mu c - cA_\mu\partial_\mu\bar{c})$$

$$+\frac{g}{8}\tilde{Z}_1^{HP}(Z_2^{HP}\tilde{Z}_2^{HP})^{-1/2}(c\partial_\mu\bar{c}A_\mu + A_\mu c\partial_\mu\bar{c} - \partial_\mu\bar{c}cA_\mu - A_\mu\partial_\mu\bar{c}c) \tag{3.16}$$

where

$$Z_1^{HP} = 1 + \frac{g^2}{16\pi^2}(-\alpha)\ln\Lambda \tag{3.17}$$

$$Z_2^{HP} = 1 + \frac{g^2}{16\pi^2}(\frac{13}{3} - \alpha)\ln\Lambda \tag{3.18}$$

$$\tilde{Z}_1^{HP} = 1, \quad \tilde{Z}_2^{HP} = 1 + \frac{g^2}{16\pi^2}(\frac{3}{2} - \frac{\alpha}{2})\ln\Lambda \tag{3.19}$$

Z_2 factors enter in the nonstandard way since only two legs (the first and the last) may bring the wave function renormalization. So we have a modification in the definition of beta function

$$\beta^{HP} = -g\frac{\partial}{\partial(\ln\Lambda)}(\frac{Z_2^{HP}}{Z_1^{HP}}) \tag{3.20}$$

Therefore we have

$$\beta^{HP} = -\frac{g^2}{16\pi^2}(\frac{13}{3} - \alpha + \alpha) = -\frac{g^2}{16\pi^2}\frac{13}{3} \tag{3.21}$$

Recall that the usual beta function is $\beta = -\frac{g^2}{16\pi^2}\frac{11}{3}$. Note that we get good results: beta remains negative and in this approximation it does not depend on the gauge fixing parameter α, i.e. it is gauge invariant.

4   Concluding remarks

In conclusion, models of quantum field theory with interaction in the Boltzmannian Fock space have been considered.

To define the master field for large N matrix models we used the Yang-Feldman equation with a free field quantized in the Boltzmannian Fock space. The master field for gauge theory does not take values in a finite-dimensional Lie algebra however there is a non-Abelian gauge symmetry. For the construction of the master field it was essential to work in Minkowski space-time and to use the Wightman correlation functions. The fact that the master field satisfies the same equations as usual relativistic fields push us to develop a non-perturbative methods of investigation of these equations. Note in this context that in all previous attempts of approximated treatment of planar theory were used some non-perturbative approximation [59, 60, 61].

To sum up a part of planar diagrams we have used the new interaction representation with a rational function of the interaction Lagrangian instead of the exponential function in the standard interaction representation. The Schwinger-Dyson equations for the 2- and 4-point correlation functions for this theory form a closed system of equations. The solutions of these equations may be used to sum up a more rich class of planar diagrams. This is a subject of further investigations.

ACKNOWLEDGMENT

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